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Teleportation of geometric structures in 3D

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Abstract

The simplest quantum teleportation algorithms can be represented in geometric terms in spaces of dimensions 3 (for real state vectors) and 4 (for complex state vectors). The geometric representation is based on geometric-algebra coding, a geometric alternative to the tensor-product coding typical of quantum mechanics. We discuss all the elementary ingredients of the geometric version of the algorithm: geometric analogs of states and controlled Pauli gates. A fully geometric presentation is possible if one employs a nonstandard representation of directed magnitudes, formulated in terms of colors defined via stereographic projection of a color wheel, and not by means of directed volumes.

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(Some figures in this article are in colour only in the electronic version)

1. Multivector geometry in 3D

The fact that vector quantities can be interpreted geometrically in *at least* two different ways was already clear to Grassmann [1], some 40 years before Gibbs [2] and Heaviside [3] invented vector calculus. One of the interpretations, close to what we are now accustomed to [4], treated vector a as a directed line segment. Grassmann introduced the *outer product* $\wedge$ that allowed us to extend two directed line segments into directed plane segments or directed line and plane segments into directed volume segments (hence, probably the name *linear extension theory* he gave to his formalism [1]). The second Grassmann interpretation treated a as a geometric point; $a \wedge b$ was a directed line segment determined by points a and b, and $a \wedge b \wedge c$ was a directed plane segment determined by three points [5]. In addition to the outer product $a \wedge b$, he introduced the inner product $a \cdot b$ acting, in a sense, in a way opposite to that of $a \wedge b$.

The two interpretations were not the only ones one could imagine. A variant of Grassmann's first interpretation (scalar and vector products) was used by Gibbs and Heaviside in their reformulation of Maxwell's electrodynamics. The two products are non-associative and define objects of different types (scalars $a \cdot b$ and pseudovectors $a \times b$, respectively), and any student knows that one should not mix them with each other. It is interesting, however, that Grassmann himself *did* contemplate a combination $\lambda a \cdot b + \mu a \wedge b$, with arbitrary nonzero constants λ, μ, and termed it the *central product*. It was Clifford who finally realized that a central product with $\lambda = \mu = 1$ defines an operation which is indeed central to the algebra of vectors [6]. Clifford's *geometric product* $ab = a \cdot b + a \wedge b$ is associative and reconstructs the two products of Grassmann by $a \cdot b = \frac{1}{2}(ab + ba)$ and $a \wedge b = \frac{1}{2}(ab - ba)$.

The Grassmann–Clifford vector calculus is completely counterintuitive for all those who learned the Gibbs–Heaviside formalism at school, but there are reasons to believe that it was Gibbs and Heaviside who spoiled the work. Perhaps the most difficult conceptual element of the geometric product is that it mixes objects of apparently different species—scalars and bivectors. But the problem is yet deeper since associativity allows us to discuss products of arbitrary numbers of vectors, leading to combinations of all the four types of 3D objects—scalars (directed points), vectors (directed line segments), bivectors (directed plane segments) and trivectors (directed volumes). Such general combinations are called polyvectors [7, 8] or multivectors [9].

Any directed line segment can be regarded as containing *two* types of directed objects of different dimensionalities: the one-dimensional interior and the zero-dimensional endpoints. The property is so obvious ('every stick has two ends') that it does not, per se, deserve further comments. However, the subtlety we want to point out is that when it comes to the *directed magnitudes* themselves, it is by no means obvious that the interior should be equipped with the same directed value as the endpoints. The magnitude of the interior of a segment is typically identified with its *length*, and if we equip the segment with a kind of arrow we obtain an interpretation of its directed value. The procedure is no longer so natural if we turn to the endpoints, and thus in what follows we prefer to think of directed magnitudes in terms of *colors* (see below for a precise mathematical definition of what we mean by this statement).

The example of the 1D segment illustrates the first idea we will develop in this paper: multivectors in 3D will be regarded as colored cubes of a fixed (e.g. unit) size, whose interiors, walls, edges and corners have colors that can differ from one another. So the basic 3D shapes (cubic interiors, square walls, segments forming the edges and the points where the edges meet) play the role of *blades* (Clifford geometric products of mutually orthonormal basis vectors) and the colors are the corresponding directed magnitudes. This type of geometric interpretation has an additional advantage of showing that a multivector is a single object whose different components are as inseparable from one another as the ends which cannot be separated from the stick.

The second goal of this paper is to show that multivectors in 3D allow for geometric implementation of the quantum teleportation protocol [10] entirely at the geometric level and without any reference to quantum mechanics. The fact that it is formally possible is a trivial consequence of two facts. First, as recently shown in [11–15], all quantum algorithms can be represented geometrically if one replaces n-bit entangled states from a 2^n-dimensional complex Hilbert space by multivectors based on a Clifford algebra of some n-, $(n + 1)$- or $(n+2)$-dimensional (Euclidean or pseudo-Euclidean) space. Second, the simplest teleportation protocol is an example of a 3 bit quantum algorithm involving only real numbers. As such, it allows for a natural geometric representation in 3D, and thus is especially attractive from the point of view of geometric representations. Continuing in a similar vein, one can extend the idea to a 3D lattice whose single cell is described by a single point, three edges, three walls and one interior—together $8 = 2^3$ basic elements typical of three dimensions—but then one needs (at least) one more natural number to characterize the cell. The full algorithm involving

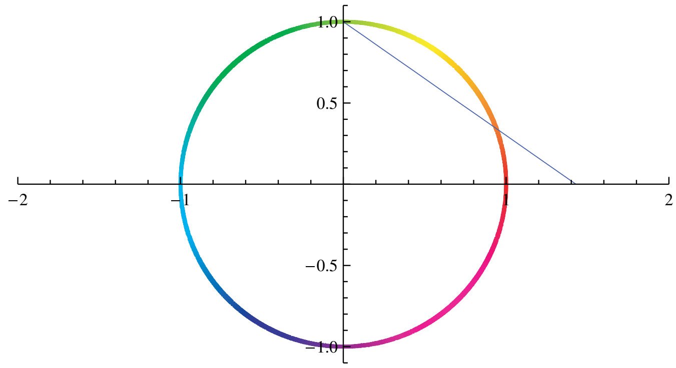

Figure 1. Stereographic projection of the hue color wheel is a one-to-one map between real numbers and hue of visible colors. If x is a real number, then the hue $h(x)$ of the color is computed in Mathematica according to $h(x) = \texttt{Hue}[\nu(x)]$, where $0 \leqslant \nu(x) < 1$ is defined implicitly by $x(1 - \sin 2\pi\nu) = \cos 2\pi\nu$.

complex amplitudes can be represented in geometric terms in 4D. All such algorithms are 'spacetime codes', a terminology used by Finkelstein [16].

2. Geometric-product coding

Consider an n-dimensional real Euclidean space and denote its orthonormal basis vectors by $b_k, 1 \leqslant k \leqslant n$. A normalized *blade* is defined by $b_{k_1\ldots k_j} = b_{k_1}\ldots b_{k_j}$, where $k_1 < k_2 < \cdots < k_j$. The basis vectors (one-blades) satisfy Clifford's geometric algebra (GA)

$$b_k \cdot b_l = \delta_{kl} = \tfrac{1}{2}(b_k b_l + b_l b_k). \tag{1}$$

The link between a binary number $A_1 \ldots A_n$ and blades (A's are bits) is given by the formula

$$c_{A_1\ldots A_n} = b_1^{A_1}\ldots b_n^{A_n}, \tag{2}$$

where it is understood that $b_k^0 = 1$. The blades $c_{A_1\ldots A_n}$ parametrized by binary sequences are occasionally referred to as combs. Sometimes one needs complex numbers; their geometric-algebra analogs can be defined in several ways (cf [13]) but in the context of teleportation one deals with gates that are real, so for simplicity we skip this point.

Let ψ be a general multivector in 3D:

$$\psi = \sum_{A,B,C=0}^{1} \psi_{ABC} c_{ABC}, \tag{3}$$

where ψ_{ABC} are real numbers. Linking ψ_{ABC} with colors by means of the stereographic projection of a color wheel[3] shown in figure 1, we obtain a geometric representation of ψ whose special case is shown in figure 2.

[3] A circle representation of colors was introduced already by Isaac Newton (Newton's color wheel) in his *Optics* (1706). Other color wheels are associated with the names of Hoener, Munsell and Ostwald; cf [17]. We employ the *hue color wheel*, discussed in detail in D Briggs' *The Dimensions of Colour*, available at http://www.huevaluechroma.com.

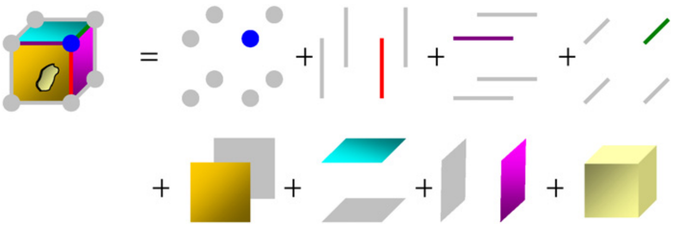

Figure 2. Example of a general 3 bit multivector $\psi = \sum_{ABC=0}^{1} \psi_{ABC} c_{ABC}$. Values of the components ψ_{ABC} can be deduced by means of the color wheel. Here we find approximately $\psi_{000} = -0.07$, $\psi_{100} = 0.32$, $\psi_{010} = -3.08$, $\psi_{001} = 1.06$, $\psi_{110} = -0.85$, $\psi_{101} = 0.27$, $\psi_{011} = -0.86$, $\psi_{111} = 4.07$. Since $h(0) = \mathtt{Hue}[3/4]$ the multivector should be, perhaps, shown on a dark-blue background corresponding to $\mathtt{Hue}[3/4]$, making invisible all elements with $x = 0$ (but then the blue corner of the cube, $\psi_{000}c_{000}$, would practically disappear from the figure). The coloring method is applicable to a cubic lattice and not only to a single cube.

3. Geometric gates and teleportation

The teleportation protocol can be described in various ways, also in purely spacetime 2-spinor terms [18]. The form which is especially useful here is the formulation in terms of a network of elementary gates acting on an initial state [19]. In the standard quantum mechanical version, one begins with the state

$$|\psi_1\rangle = \alpha|0_1\rangle + \beta|1_1\rangle, \tag{4}$$

which is to be teleported, and the entangled state,

$$|\Phi_{23}\rangle = \frac{1}{\sqrt{2}}(|0_2 0_3\rangle + |1_2 1_3\rangle), \tag{5}$$

which plays the role of a carrier of quantum information, and is one of the four 2 bit entangled states forming the so-called Bell basis (figure 3). The Bell basis can be regarded as an analog of the Minkowski tetrad [20], if one translates qubits into 2-spinors [18] and $|\Phi_{23}\rangle$ is then an analog of the spacelike worldvector x^a [18]. The protocol does not need the concrete state $|\Phi_{23}\rangle$, but any non-factorizable 2 bit state can be employed—the 2-spinor protocols analyzed in [18] employ analogs of y^a and ε^{AB}.

The goal is to implement the map

$$|\psi_1\rangle = \alpha|0_1\rangle + \beta|1_1\rangle \to |\psi_3\rangle = \alpha|0_3\rangle + \beta|1_3\rangle \tag{6}$$

with unknown α, β. The network of gates acts as follows:

$$H_1 H_2 Z_3{}^1 X_3{}^2 H_1 X_2{}^1 |\psi_1\rangle|\Phi_{23}\rangle = |0_1 0_2\rangle(\alpha|0_3\rangle + \beta|1_3\rangle), \tag{7}$$

where X_k, Z_k, $H_k = (X_k + Z_k)/\sqrt{2}$ are the Pauli X (the NOT gate) and Z, and Hadamard gates acting on kth bits; $X_k{}^l$, $Z_k{}^l$ are the Pauli gates acting on kth bits and controlled by lth bits. Below we shall give their explicit definition already in a geometric form, so let us first explain the geometric analog of teleportation. We begin with the multivectors

$$\psi_1 = \alpha c_{0_1} + \beta c_{1_1} = \alpha + \beta b_1, \tag{8}$$

$$\Phi_{23} = \frac{1}{\sqrt{2}}(c_{0_2 0_3} + c_{1_2 1_3}) = \frac{1}{\sqrt{2}}(1 + b_2 b_3). \tag{9}$$

The teleportation network must therefore act as follows:

$$\begin{aligned} H_1 H_2 Z_3{}^1 X_3{}^2 H_1 X_2{}^1 \psi_1 \Phi_{23} &= c_{0_1 0_2}(\alpha c_{0_3} + \beta c_{1_3}), \\ &= \alpha + \beta b_3. \end{aligned} \tag{10}$$

The elementary geometric gates act in direct analogy to their quantum counterparts. Below, we list the nontrivial actions of the Pauli gates:

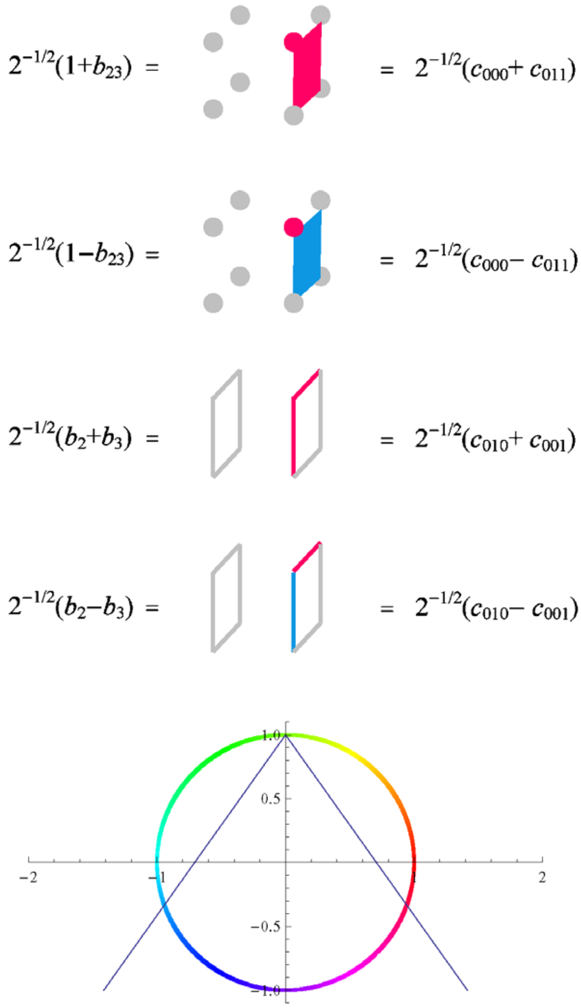

Figure 3. Multivector analogs of the Bell basis of the last 2 bit. The color wheel is used to identify the colors corresponding to $\pm 1/\sqrt{2} \approx \pm 0.71$ (`Hue`[0.554] and `Hue`[0.946]).

$$
\begin{aligned}
X_2{}^1 c_{100} &= c_{110},\\
X_2{}^1 c_{101} &= c_{111},\\
X_2{}^1 c_{110} &= c_{100},\\
X_2{}^1 c_{111} &= c_{101},\\
X_3{}^2 c_{010} &= c_{011},\\
X_3{}^2 c_{011} &= c_{010},\\
X_3{}^2 c_{110} &= c_{111},\\
X_3{}^2 c_{111} &= c_{110},\\
Z_3{}^1 c_{100} &= c_{100},\\
Z_3{}^1 c_{101} &= -c_{101},
\end{aligned}
$$

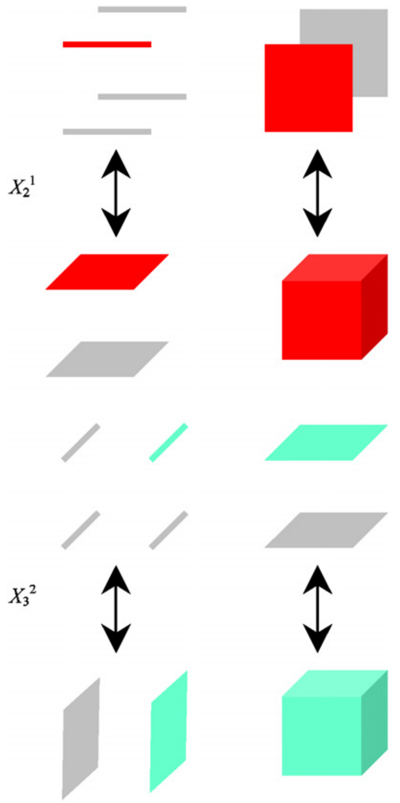

Figure 4. Controlled X's. Only the blades containing b_1 are affected by ${X_2}^1$ (b_1 is the edge parallel to the x-axis, $b_{12} = b_1 b_2$ is the unit square in the x–y plane and $b_{123} = b_1 b_2 b_3$ is the unit cube). Similarly, only the blades that contain b_2 are affected by ${X_3}^2$. The gates act trivially on the remaining blades.

$$
\begin{aligned}
{Z_3}^1 c_{110} &= c_{110}, \\
{Z_3}^1 c_{111} &= -c_{111}, \\
X_1 c_{1BC} &= c_{0BC}, \\
X_1 c_{0BC} &= c_{1BC}, \\
X_2 c_{A1C} &= c_{A0C}, \\
X_2 c_{A0C} &= c_{A1C}, \\
Z_1 c_{1BC} &= -c_{1BC}, \\
Z_2 c_{A1C} &= -c_{A1C}.
\end{aligned}
$$

Translating these formulae into the language of blades, we arrive at the following nontrivial actions of the controlled gates:

$$
\begin{aligned}
& b_1 \overset{{X_2}^1}{\leftrightarrow} b_{12}, \\
& b_{13} \overset{{X_2}^1}{\leftrightarrow} b_{123}, \\
& b_2 \overset{{X_3}^2}{\leftrightarrow} b_{23},
\end{aligned}
$$

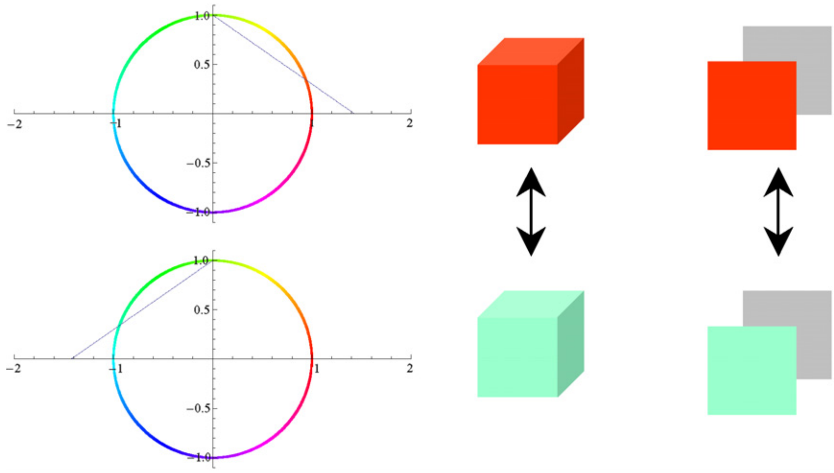

Figure 5. ${Z_3}^1$ affects only those blades that contain b_1 (then the controlling first bit equals 1) and b_3. The gate changes color of the blade according to $h(x) \to h(-x)$.

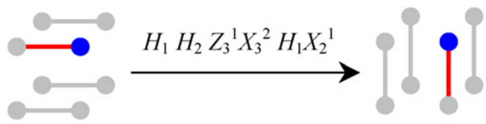

Figure 6. The effect of the teleportation protocol on a multivector $\alpha + \beta b_1$.

$$
\begin{aligned}
&b_{12} \overset{{X_3}^2}{\leftrightarrow} b_{123},\\
&b_{13} \overset{{Z_3}^1}{\leftrightarrow} -b_{13},\\
&b_{123} \overset{{Z_3}^1}{\leftrightarrow} -b_{123}.
\end{aligned}
$$

The gates X_k create or annihilate the basis vector b_k in a blade (i.e. expand or squeeze the blade along the kth direction) and Z_k change the sign of blade if b_k is present (i.e. appropriately change the color of blades containing b_k). Figures 4 and 5 show the geometry of the controlled Pauli gates. The Hadamard gates are a combination of the two actions. Figure 6 shows the end result of the teleportation protocol.

The cubes in the above examples are colored in a way that allows for merging them into cubic lattices. Each cube has to be equipped with its own GA. In 1D we would have pairs of blades $\{1_k, b_k\}$, $k = 0, \pm1, \pm2, \ldots,$ that are canonically isomorphic to a single GA with blades $\{1, b\}$; in 2D we have $\{1_{kl}, b_{1kl}, b_{2kl}, b_{12kl}\}$, $k, l = 0, \pm1, \pm2, \ldots,$ canonically isomorphic to $\{1, b_1, b_2, b_{12}\}$ and so on. Any quantum GA protocol will influence each of the cells individually, and thus will play a role of an internal symmetry transformation. No change of interpretation is needed if one generalizes these geometric structures to curvilinear lattices (color-preserving deformations of cubic lattices).

A natural geometric arena for geometric analogs of quantum teleportation is provided by 3D or 4D lattices, whose basic cells can be regarded as multivectors of dimension 2^3 or 2^4, respectively. It would be interesting to consider the limiting case of a continuous-space limit of a multivector lattice and the corresponding field theory (cf the attempts of formulating field theory on the Clifford space of points, areas and volumes, proposed by Pavšič [7, 8]).

Although in the present paper we work only with the GA of 3D Euclidean spaces, the transition to Minkowski space and more general Lorentzian manifolds is immediate [21].

Finally, let us remark that the interpretation of oriented magnitudes in terms of colors allows for visualizations of yet higher dimensional geometric structures. The point is that the space of colors is in fact at least three dimensional (the dimensions are known as hue, chromaticness and brightness) [22]. These dimensions are typically regarded as being compact but, as we have seen with the example of the hue, they may be regarded as compactified forms of non-compact ones. So the approach we have outlined above naturally extends beyond 3D and 4D, and may lead to interdisciplinary applications that go much beyond standard physics.

Acknowledgments

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